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COMMENT

Scaling properties of diffusion on comb-like structures

Ofer Matan†, Shlomo Havlin† and Dietrich Stauffer‡

† Department of Physics, Bar-Ilan University, Ramat-Gan, Israel
‡ HLRZ, c/o KFA Julich, PO Box 1913 Julich 1, West Germany

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Abstract. We study the scaling properties of the probability density of random walks on comb-like structures. We find that, in contrast to the first-passage time, in which several exponents are needed to characterise their distribution, the probability density can be described by a single exponent.

The scaling properties of the density distribution of diffusion on disordered structures and fractals structures have recently been extensively studied [1-5]. In a recent work [2] the first-passage time (FPT) distribution, $S_L(t)$, for diffusion on a family of hierarchical comb structures (see figure 1) was studied. Here t represents the time needed for a random walker to reach for the first time a distance L along the backbone of the comb. It was shown that $S_L(t)$ do not scale as a function of $t/\langle t\rangle$, where $\langle t\rangle$ is the mean FPT. At least two different exponents which are needed to characterise the distribution were identified. One characterises the scaling of the mean first-passage time $\langle t\rangle$ as a function of L, $\langle t\rangle \sim L^{\tau_1}$. The other characterises the scaling of the high first-passage time moments as a function of L, $\langle t^q\rangle \sim L^{\tau_q}$. The values of these exponents are [3, 4]

$$\tau_1 = 1 + \frac{\ln R}{\ln 2} \tag{1a}$$

$$\tau_q = 1 + (2q-1)\frac{\ln R}{\ln 2} \qquad \tau^* \equiv \frac{d\tau_q}{dq} = 2\frac{\ln R}{\ln 2} \qquad q \geqslant 1. \tag{1b}$$

The parameter R characterises the size of the teeth in the hierarchical comb (see figure 1). This suggests that the first-passage time distribution does not scale as a function of $t/\langle t\rangle$ and is of a multifractal nature. The deviations from scaling are seen in figure 2. Indications for other exponents representing the non-integer moments with q smaller than 1 were observed numerically [2].

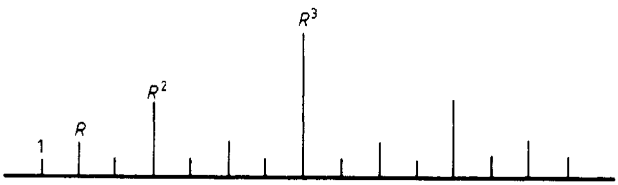

Figure 1. Hierarchical comb structure. The random walker makes unit steps on the backbone and the teeth of the structure. The third generation of the hierarchical comb is shown here.

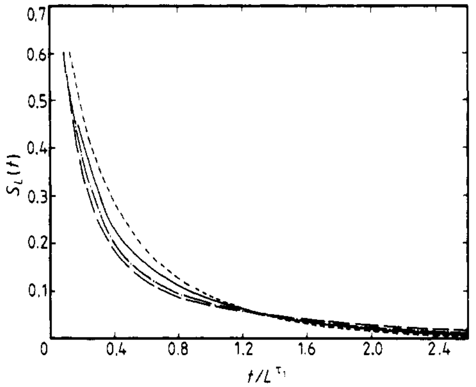

Figure 2. The survival probability is plotted for $R = 3$ for different sizes of the hierarchical comb: $L = 7$ (short broken curve), $L = 15$ (full curve), $L = 31$ (chained curve) and $L = 63$ (long broken curve). The survival probability is plotted as a function of the trial scaling variable t/L^{τ_1}. Scaling in different regimes is achieved by using different powers of L. The non-scaling property of the function for all times leads to the multifractality of the exponents τ_q (see figure 3 of [2]).

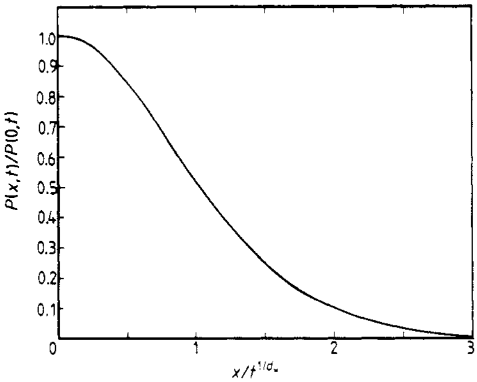

Figure 3. The normalised probability density function on the backbone $P(x, t)$ for $R = 3$ is plotted as a function of the scaling variable $x/t^{1/d_w}$. The function is plotted for different times (20 000, 30 000 and 40 000 time steps). The curves collapse and are indistinguishable on this scale. Note that, in contrast to $S_L(t)$, the probability density $P(x, t)$ scales for all length and time scales.

We present here numerical data suggesting that, in contrast to the behaviour of $S_L(t)$, the probability density $P(x, t)$ of a random walker to be at distance x at time t on hierarchical combs can be presented in a scaling form

$$P(x, t) \sim t^{-d_s/2} g(x/t^{1/d_w}). \tag{2}$$

Moreover, the exponent d_w was found to be $d_w = 4 \ln R/\ln 2R$, and is different from the two exponents τ_1 and τ^* which appear in the FPT problem. In other words, while 'time as a function of distance' seems multifractal, 'distance as a function of time' scales with a single exponent.

We studied diffusion on the hierarchical combs using the exact enumeration method [1]. The method yields exact numerical values for $P(x, t)$. Results for $P(x, t)t^{d_s/2}$ as a function of $x/t^{1/d_w}$ are shown in figure 3. It is seen that, in contrast to figure 1, all curves collapse to a single curve. We also calculated the moments $\langle x^q \rangle$ as a function of t. Assuming that

$$\langle x^q \rangle \sim t^{1/d_w(q)} \tag{3}$$

we plot, in figure 4, $1/d_w(q)$ as a function of q. It is seen, within the accuracy of our numerical data, that $1/d_w(q)$ is *proportional* to q for all calculated values of q. The slope of figure 4 is in excellent agreement with

$$\frac{1}{d_w(q)} = \frac{q}{d_w} = q\frac{\ln 2R}{4 \ln R}. \tag{4}$$

This result strongly supports the scaling form of (2). This is in clear contrast with the moments of the FPT where non-linear behaviour appears in the range of small FPT moments (see figure 3 of [2]). For several systems, such as percolation and several fractals, the probability density of a random walker $P(r, t)$ to be in site r at time t

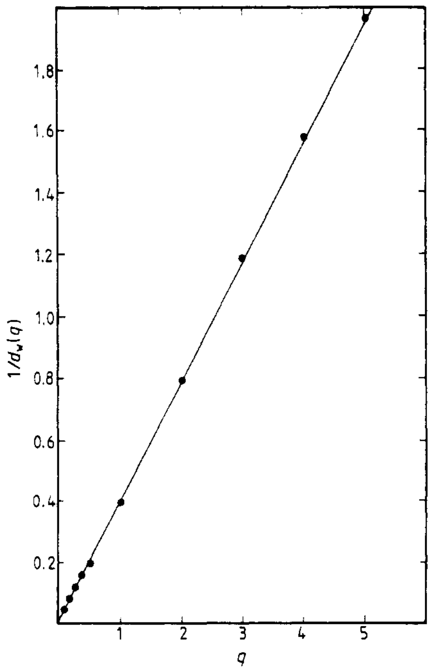

Figure 4. The exponents $1/d_w(q)$ are plotted as a function of q. The function is proportional to q, implying the scaling property of the density distribution $P(x, t)$.

when starting at $t = 0$ from the origin, was found, numerically to have a scaling form similar to (2). This scaling function contributes to a linear spectrum of moments of the displacement as a function of time. It would be interesting to study small moments of FPT on such system in order to explore the possibility of the existence of new diffusion exponents.

In summary, the distribution of times for fixed distance can be quite different to the distribution of distances for fixed time, as also observed elsewhere [5]. This difference may be due to extremely rare events [6].

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